I. Sinusoids, Exponentials, and Electrical Circuits

A. Introduction

It is a remarkable fact of nature that almost any homogeneous and stationary physical system when lightly perturbed will decompose that perturbation as it propagates through the system into sinusoids that will move at different speeds, while decaying exponentially. Homogeneous and stationary systems are systems that do not change in space or time. This remarkable fact of nature is true when you throw a rock in a pond; after an initial splash, sinusoids will propagate outward. It is true for for pressure waves moving through the earth or through the scaffolding of a building or a bridge. It is true for sound waves. It is true for the vibrational waves on a violin string. It is true for radio waves propagating in the earth-ionosphere waveguide and for light waves propagating through optical fibers. It is true for gravity waves propagating through the interstellar medium. It is even true for the probability waves that govern the positions of electrons and holes in solids.

It is this fact that makes sinusoids and their properties so important in the study of physical systems and engineering.

Of course, no real system is exactly homogeneous and stationary. Moreover, we can question how “lightly” it is necessary to perturb a real system to generate sinusoids. However, these approximations work well enough in real systems that they are the starting point for analyzing a vast array of systems, including the electrical and photonic systems that are our own interest.

This remarkable fact of nature is closely allied to the mathematical fact that the solution of any linear ordinary differential equation with constant coefficients can be written as a sum of complex exponential, perhaps multiplied by polynomials.

The goal of this review is to cover the basic mathematical tools that are needed to analyze linear circuits and more generally *any* linear system.

B. Sine and Cosine Functions

Sine functions can be defined geometrically using the unit circle. Given any point (x,y) on the unit circle, we define the sine function as $\sin(\theta) = y(\theta)$ and the cosine function as $\cos(\theta) = x(\theta)$, where θ is the angle in radians, as we show in Fig. 1.

In Fig. 2, we show plots of the sine and cosine functions as θ changes between 0 and 6π.

These functions repeat periodically every 2π. Given the definition of the cosine and sine functions, it follows that

$$\begin{aligned} \cos(-\theta) &= \cos(\theta), \quad \sin(-\theta) = -\sin(\theta) \\ \cos(\theta+\pi/2) &= -\sin(\theta), \quad \sin(\theta+\pi/2) = \cos(\theta) \end{aligned} \tag{1}$$

From the definition, it also follows that $\cos^2\theta + \sin^2\theta = 1$.

We can now derive the angle addition and subtraction formulae. We consider two angles α and β as shown in the first half of Fig. 3. The distance d between the two points (x_2, y_2) and (x_1, y_1) is the same as the distance d between the point (x_c, y_c), defined by the angle $\beta - \alpha$, shown on the right, and the point $(1,0)$ on the x-axis of the unit circle. We have just rotated the circle by $-\alpha$.

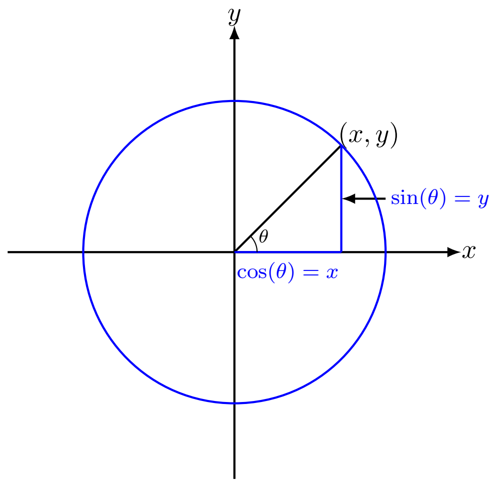

Figure I.1

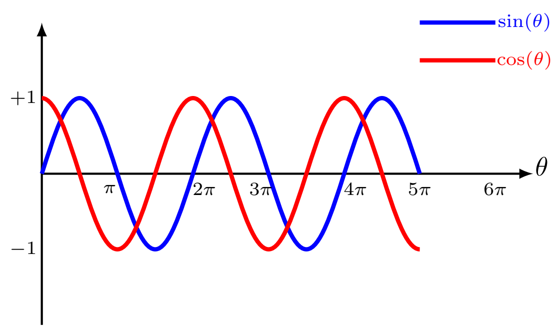

Figure I.2

We now find

$$\begin{aligned} d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 = 2 - 2\cos\beta\cos\alpha - \sin\alpha\sin\beta \\ &= (x_c - 1)^2 + y_c^2 = 2 - 2\cos(\beta - \alpha), \end{aligned} \tag{2}$$

from which it follows

$$\cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha. \tag{3}$$

Using Eq. (1), we can now find by substitution

$$\begin{aligned} \cos(\beta + \alpha) &= \cos\beta\cos\alpha - \sin\beta\sin\alpha, \\ \sin(\beta - \alpha) &= \sin\beta\cos\alpha - \cos\beta\sin\alpha, \\ \sin(\beta + \alpha) &= \sin\beta\cos\alpha + \cos\beta\sin\alpha. \end{aligned} \tag{4}$$

We will also need the relation

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1. \tag{5}$$

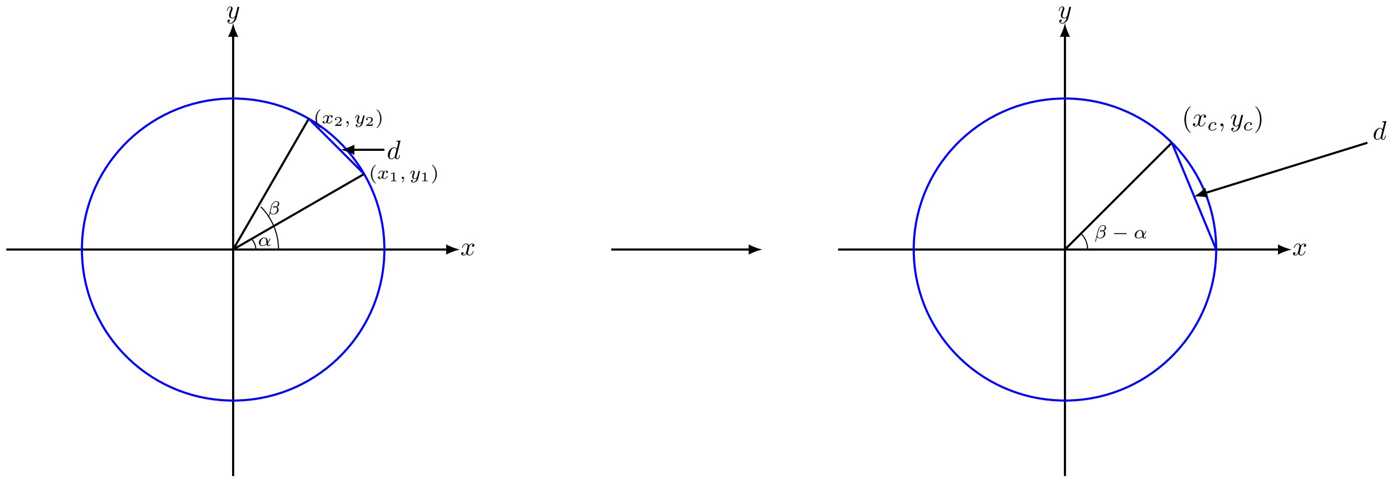

Figure I.3

To obtain the relationship, we consider the circular sector subtended by the angle θ that we show in Fig. 4. The area of the sector is $\theta/2$; the area of the inner triangle is $(1/2)\sin\theta\cos\theta$; and the area of the outer triangle is $(1/2)(\sin\theta/\cos\theta)$. It follows that

$$\sin\theta\cos\theta < \theta < \sin\theta/\cos\theta, \tag{6}$$

and noting that $\cos\theta \to 1$ when $\theta \to 0$, we obtain the final result. This last derivation is our last geometric derivation. From hereon, we proceed algebraically.

We can now derive the derivative relations from the basic definition of the derivative. We have

$$\begin{aligned} \frac{d\sin\theta}{d\theta} &= \lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}[\sin(\theta+\Delta\theta/2) - \sin(\theta-\Delta\theta/2)] \\ &= \lim_{\Delta\theta\to 0}\frac{2}{\Delta\theta}\cos\theta\sin(\Delta\theta/2) = \cos\theta. \end{aligned} \tag{7}$$

It follows immediately that $d\cos\theta/d\theta = -\sin\theta$. From the fundamental theorem of calculus (not proved here), the integral is given by the inverse of the derivative. We thus find

$$\int \sin\theta\, d\theta = -\cos\theta + C, \qquad \int \cos\theta\, d\theta = \sin\theta + C, \tag{8}$$

where C indicates an arbitrary constant that is determined by the integration limits.

C. Complex Numbers and Exponentials

Complex numbers are defined as $z = x + iy$, where $i = \sqrt{-1}$, while x and y are real numbers. Complex numbers can be represented in the (x, y) plane—usually referred to as the complex plane in this context, as we show in Fig. 5.

We see that the complex numbers can also be represented as $z = r\cos\theta + ir\sin\theta$, where $r = (x^2 + y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$. This second representation is referred to as the polar representation.

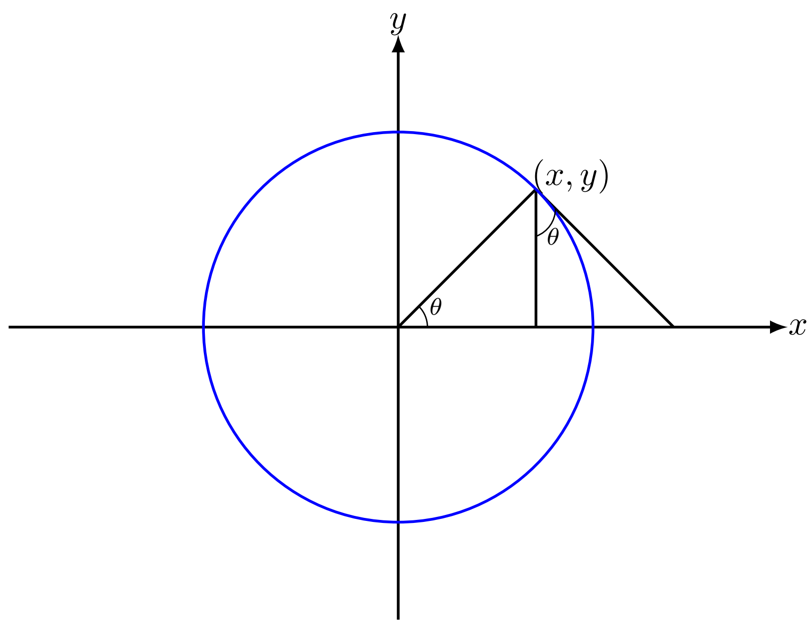

Figure I.4

Because the sine and cosine functions are periodic, this representation is not unique. The number $z_n = r\cos(\theta + 2\pi n) + ir\sin(\theta + 2\pi n)$ is the same when n equals any integer value.

Addition and subtraction of complex numbers are defined as follows

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), \qquad z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2). \tag{9}$$

Multiplication can be defined using the distributive relation

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1). \tag{10}$$

To define division, we first define the complex conjugate, $z^* = x - iy$. We then find

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{|z_2|^2} = \frac{(x_1 x_2 + y_1 y_2) - i(x_1 y_2 - x_2 y_1)}{(x_2^2 + y_2^2)}. \tag{11}$$

We now define the exponential function, which will see greatly simplifies the discussion of multiplication, division, and much else.

We let $\exp(i\theta) = \cos\theta + i\sin(\theta)$. We see that as θ increases, the values of $\exp(i\theta)$ trace out the unit circle in the complex plane. We can show the important relation

$$\exp[i(\theta_1 + \theta_2)] = \exp(i\theta_1)\exp(i\theta_2), \tag{12}$$

which follows from the angle addition and subtraction formulae. We find

$$\begin{aligned} \exp(i\theta_1)\exp(i\theta_2) &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = \exp[i(\theta_1 + \theta_2)]. \end{aligned} \tag{13}$$

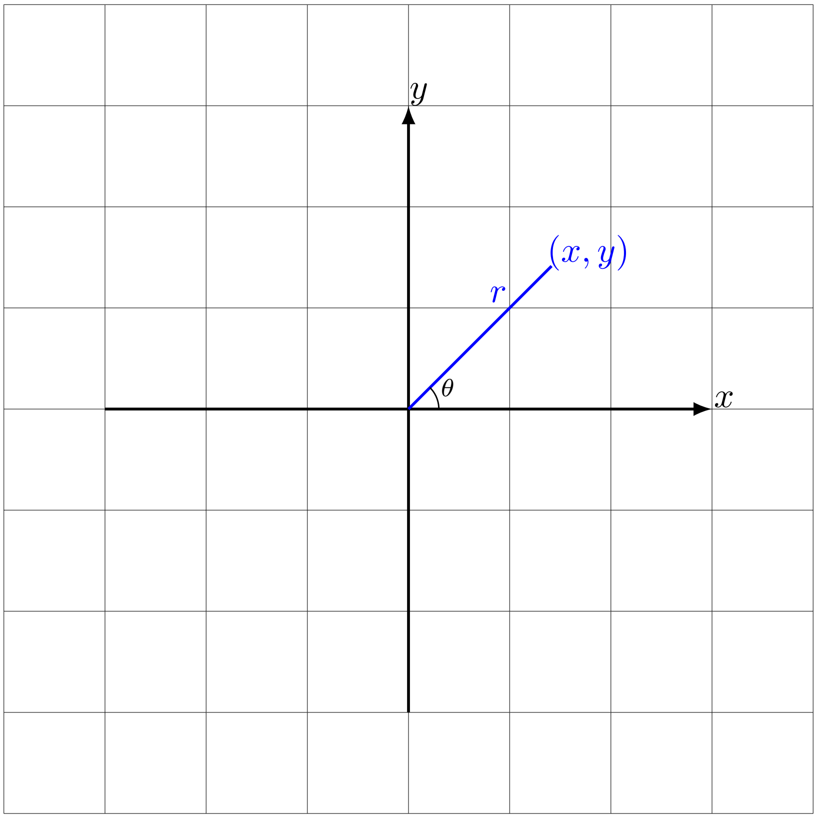

Figure I.5

In terms of the complex expontial, we have

$$z_1 z_2 = r_1 r_2 \exp[i(\theta_1 + \theta_2)], \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} \exp[i(\theta_1 - \theta_2)]. \tag{14}$$

We can now easily define powers and roots. We have

$$z^n = r^n \exp(in\theta), \qquad z^{1/n} = r^{1/n} \exp[i(1/n)(\theta + 2\pi m)], \tag{15}$$

where m and n are integers. In the complex plane, there are n independent solutions for the n-th root. So, for example, the number 27 has as its cube roots 3, $3\exp(2\pi i/3)$, and $3\exp(-2\pi i/3)$. Finally, we note the relations

$$\cos\theta = \frac{1}{2}[\exp(i\theta) + \exp(-i\theta)], \qquad \sin\theta = \frac{1}{2i}[\exp(i\theta) - \exp(-i\theta)]. \tag{16}$$

Another important property of the complex exponential is its derivative. Using the derivative relations for the cosine and sine, we find

$$\frac{d\exp(i\theta)}{d\theta} = i\exp(i\theta), \tag{17}$$

so that the derivative of the complex exponential just equals itself multiplied by the factor i.

We now want to extend the definition of the exponential function so that its arguments can be complex. To do that, it is useful to introduce the Taylor expansions of the exponential function at $x = 0$. We recall the Taylor expansion formula (not proved here) is given by

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}, \tag{18}$$

where $f^{(n)} = df^n(x)/dx|_0$ is the n-th derivative of the function evaluated at zero. We now find that

$$\exp(i\theta) = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}. \tag{19}$$

For an arbitrary complex number z, we now let

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{20}$$

This sum converges for any z, so that this definition of the exponential is well-defined for any z. We now find

$$\frac{d\exp z}{dz} = \exp z \tag{21}$$

and

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2). \tag{22}$$

It follows that

$$\exp(nz) = [\exp z]^n = [\exp(x)]^n \exp(iny), \tag{23}$$

where n is an integer. We see that this corresponds to multiplying the magnitude by a constant multiple for each increase of n by 1 and rotating the phase angle by a constant amount. For an oscillator circuit, we will see that this relationship translates into a decrease by a fixed factor over every interval of time and a rotation of the output phase by a fixed amount.

D. Ordinary Differential Equations and Phasors

Any linear system without an external energy source has natural frequencies at which it wants to oscillate, along with exponential decay. In particular, any circuit that consists of a combination of resistors, capacitors, and inductors can be described by an n-th order ordinary differential equation with constant coefficients. The homogeneous version of this equation, which is what determines the natural frequencies, may be written

$$0 = \sum_{m=0}^{n} A_m \frac{d^m u(t)}{dt^m}, \tag{24}$$

where C_n can be set equal to 1. If it is not 1 initially, we divide through by it. In a circuit, the quantity u is either the voltage or the current at a load. To find the natural frequencies, we substitute $u(t) = \exp(\gamma t)$ into Eq. (23), which becomes

$$0 = \sum_{m=0}^{n} A_m \gamma^n. \tag{25}$$

Eq. (24) is an n-th order polynomial, which has n roots γ_j, in terms of which Eq. (24) can be written as

$$0 = \prod_{j=1}^{n} (\gamma - \gamma_j). \tag{26}$$

It is evident that given Eq. (25), we can obtain the A_m. The converse is less obvious and referred to as the fundamental theorem of algebra. Obtaining the γ_j when n is large is in fact a difficult computational problem. Once the γ_j have been found, we can write the general solution to Eq. (23) as

$$u(t) = \sum_{j=1}^{n} C_j \exp(\gamma_j t), \tag{28}$$

where the values of the C_j are determined by the initial conditions, which are the values of the $d^m u/dt^m$, $m = 0, 1, \ldots, n-1$ at $t = 0$. An important caveat is that if two or more of the values of the γ_j coincide, it is necessary to some of the coefficients by polynomials. We will not be concerned with this special case here.

In circuits with resistors, inductors, and capacitors, we can say a bit more about the γ_j. In this case, the A_m will all be real, and it follows that the γ_j are purely real or come in complex conjugate pairs. Hence, we can write $\gamma_j = -\alpha_j + i\omega_j$. Moreover, the presence of resistors implies that the circuit will lose energy; so, we must have $\alpha_j > 0$. Additionally, since $u(t)$ must be real in this case, the coefficients C_j are constrained. If we let $j = 1, \ldots, p$ refer to $\gamma_j = -\alpha_j$ that are purely real, then the corresponding C_j must be real. If we let $j = p+1, \ldots, n$ refer to pairs of γ_j such that $\gamma_j = -\alpha_j + i\omega_j$, $\gamma_{j+1} = -\alpha_j - i\omega_j$, then we must have $C_{j+1} = C_j^*$, and Eq. (26) becomes

$$u(t) = \sum_{j=1}^{p} C_j \exp(-\alpha_j t) + \sum_{l=1}^{(n-p)/2} 2|C_{p+2l-1}| \cos(\omega t + \theta_{p+2l-1}) \exp(-\alpha_{p+2l-1} t), \tag{27}$$

where, writing $C_q = |C_q| \exp(i\theta_q)$, where $p + 2l - 1$, we have substituted

$$\begin{aligned} &C_q \exp(\gamma_q t) + C_{q+1} \exp(\gamma_{q+1} t) \\ &\qquad = [C_q \exp(i\omega_q t) + C_{q+1} \exp(-i\omega_{q+1} t)] \exp(-\alpha_q t) \\ &\qquad = |C_q| [\exp(i\omega_q t + i\theta_q) + \exp(-i\omega_q t - i\theta_q)] \exp(-\alpha_q t) \\ &\qquad = 2|C_q| \cos(\omega_q t + \theta_q) \exp(-\alpha_q t). \end{aligned} \tag{29}$$

The key takeaway from all this mathematics is that once a circuit is started up—and we will talk shortly about how that can be done—any load will oscillate sinusoidally with some combination of frequencies (including zero), while the voltage and/or current at the load slowly decays. As noted in the introduction, the same holds true for *any* linear system—not just circuits. So, these ideas are very important. Good oscillator circuits of the sort in which we are interested have very low attenuation, so that $\alpha_j \ll \omega_j$

We now consider the RLC circuit that we show in Fig. 6.

Before $t = 0$, switch S_1 is open and switch S_2. At that time, we have no current flow so that $V_1 = V_2 = V_0$. When switch S_2 is opened and switch S_1 is closed, the charge on the capacitor begins to discharge through the inductor and the resistor. The circuit evolution is described by the following equations

$$L\frac{dI(t)}{dt} + RI(t) - V(t) = 0, \qquad \frac{dV(t)}{dt} + \frac{I(t)}{C} = 0. \tag{30}$$

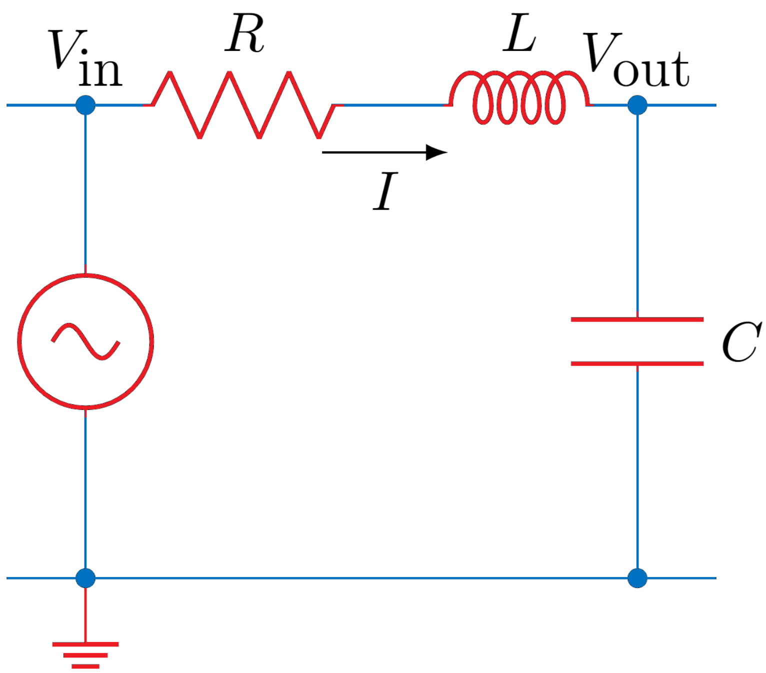

Figure I.6

These equations are in the form of two coupled first-order equations, rather than a single second-order. We can turn Eq. (29) into a second-order equation by taking the derivative of the second equation derivative and then using both equations to eliminate $I(t)$ and $dI(t)/dt$. If we do that, we find

$$\frac{d^2V(t)}{dt^2} + \frac{R}{L}\frac{dV(t)}{dt} + \frac{1}{LC}V(t) = 0. \tag{31}$$

Substituting $\exp(\gamma t)$ into this equation to find the natural frequencies, we find

$$\gamma^2 + \frac{R}{L}\gamma + \frac{1}{LC} = 0, \tag{32}$$

which implies that

$$\gamma = -\frac{R}{2L} \pm \left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)^{1/2}. \tag{33}$$

In the case of greatest interest to us, we have $(1/LC)^{1/2} \gg (R/2L^2)$, and our two natural frequencies are complex congugates with $\gamma_1 = -\alpha + i\omega$ and $\gamma_2 = -\alpha - i\omega$ with

$$\alpha = \frac{R}{2L}, \qquad \omega = \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2}. \tag{34}$$

The general solution can be written as

$$V(t) = \tilde{V}\exp(-\alpha t + i\omega t) + \tilde{V}^*\exp(-\alpha t - i\omega t), \tag{35}$$

where $\tilde{V}$ must be determined from the initial conditions. We also have

$$\frac{dV(t)}{dt} = (-\alpha + i\omega)\tilde{V}\exp(-\alpha t + i\omega t) + (-\alpha - i\omega)\tilde{V}^*\exp(-\alpha t - i\omega t). \tag{36}$$

The initial conditions are $V(t=0) = V_0$ and $dV(t)/dt|_0 = 0$. The derivative must be zero because the inductor forces the initial current flow to equal zero. We now find

$$\tilde{V} + \tilde{V}^* = V_0, \qquad (-\alpha + i\omega)\tilde{V} + (-\alpha - i\omega)\tilde{V}^* = 0. \tag{37}$$

We then obtain

$$\tilde{V} = \frac{1}{2}\left(1 - \frac{i\alpha}{\omega}\right)V_0.$$

Writing the solution as a cosine function, we then find

$$V(t) = V_0\frac{\cos(\omega t + \theta)}{\cos\theta}\exp(-\alpha t), \tag{38}$$

where $\theta = \tan^{-1}(\alpha/\omega)$. The current is then given by

$$I(t) = \omega C V_0\frac{\sin(\omega t)}{\cos\theta}\exp(-\alpha t). \tag{39}$$

We see that the current and the voltage are almost, but not exactly a phase $\pi/2$ apart, which is typical of a weakly damped oscillator.

In this case, we supposed that the oscillator was launched with a fixed initial condition and then allowed to relax at its natural frequency. The more general case is that we drive the circuit with an input signal. In the most general case, Eq. (23) becomes

$$\sum_{m=0}^{n} A_m\frac{d^m u(t)}{dt^m} = v(t), \tag{40}$$

where $v(t)$ is the driving signal. When the driving signal is periodic, write the driving signal as

$$v(t) = \frac{v_0}{2} + \sum_{l=1}^{\infty} v_l\cos(il\omega_0 t + \theta_l), \tag{41}$$

where if T is the period, then $\omega_0 = 2\pi/T$ is referred to as the fundamental angular frequency. We can rewrite this sum as

$$v(t) = \frac{1}{2}\sum_{l=-\infty} \tilde{v}_l\exp(il\omega_0 t), \tag{42}$$

where $\tilde{v}_0 = v_0$ is a constant contribution. The quantities $\tilde{v}_l$ are referred to as phasors in electrical circuit theory. They differ from the usual Fourier components by a factor two. We will defer the question of to calculate $\tilde{v}_l$, given $v(t)$ until later.

The key point is that in any linear system, the response of the system to a sinusoidal driver is a sinusoid with the same frequency. Hence, each of the terms in the driving signal can be treated independently, and the can all be summed at the end.

In the case of Eq. (39), we see that if we write

$$u(t) = \frac{1}{2}\sum_{l=-\infty}^{\infty} \tilde{u}\exp(i\omega_l t), \tag{43}$$

then it follows that

$$\tilde{u}_l = \left[\sum_{m=0}^{n} (il\omega_0)^m\right]^{-1} \tilde{v}_l \tag{44}$$

solves this equation. It is referred to as a particular solution because we can add any solution of the homogeneous equation, Eq. (23), to this particular solution, and we still have a solution to the inhomogeneous equation, Eq. (39). However, since the solutions to Eq. (23) all decay, the particular solution will ultimately become all that is left. Hence, it is often referred to as the steady-state solution, while the solutions that decay are referred to as transients. The transient contributions to the complete solution are determined by the initial conditions.

We now consider the driven RLC circuit that we show in Fig. 7.

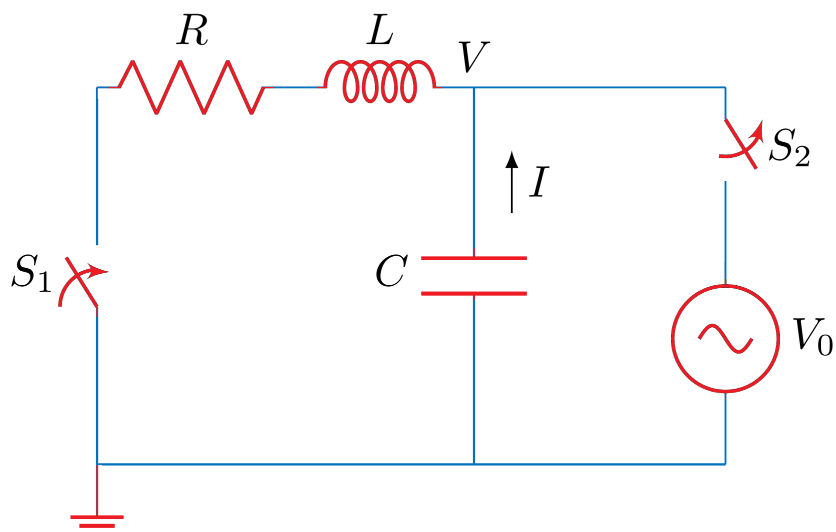

Figure I.7

This circuit is the first stage in a ladder circuit and appears in the theory of transmission lines. We will suppose that we have a single input frequency, so that the driving voltage is given by

$$V_{\rm in}(t) = V_{\rm i}\cos(\omega_0 t + \theta_{\rm i}) = \Re\left[\tilde{v}_{\rm i}\exp(i\omega_0 t)\right], \tag{45}$$

where $\tilde{v}_{\rm i} = V_{\rm i}\exp(i\theta_{\rm i})$. We wish to calculate $V_{\rm out}$. We see that this circuit functions as a voltage divider. We have the equation

$$\begin{aligned} L\frac{dI}{dt} + RI &= V_{\rm in}(t) - V_{\rm out}, \\ \frac{dV_{\rm out}(t)}{dt} &= \frac{I}{C}. \end{aligned} \tag{46}$$

We can derive a second-order ordinary differential equation for $V_{\rm out}$ by eliminating I, and we find

$$\frac{d^2V_{\rm out}}{dt} + \frac{R}{L}\frac{dV_{\rm out}}{dt} + \frac{1}{LC}V_{\rm out} = \frac{1}{LC}V_{\rm in}. \tag{47}$$

We now write

$$V_{\rm out} = V_{\rm o}\cos(\omega_0 t + \theta_{\rm o}) = \Re\left[\tilde{v}_{\rm o}\exp(i\omega_0 t)\right], \tag{48}$$

where $\tilde{v}_\mathrm{o} = V_\mathrm{o}\exp(i\theta_\mathrm{o})$. In the phasor domain, Eq. (45) becomes

$$\left(-\omega_0^2 + \frac{R}{L}i\omega_0 + \frac{1}{LC}\right)\tilde{v}_\mathrm{o} = \frac{1}{LC}\tilde{v}_\mathrm{i}. \tag{49}$$

It is convenient to write $1/LC = \omega_\mathrm{r}^2$ and $\alpha = R/2L$, where we note that ω_r is close to the angular oscillation frequency of the circuit. We then have

$$\tilde{v}_\mathrm{o} = \frac{\omega_\mathrm{r}^2[(\omega_\mathrm{r}^2-\omega_0^2)-2i\alpha\omega_0]}{(\omega_\mathrm{r}^2-\omega_0^2)^2+4\alpha^2\omega_0^2}\tilde{v}_\mathrm{i}. \tag{50}$$

Returning to the time domain, we find

$$V_\mathrm{o} = \frac{\omega_r^2}{[(\omega_\mathrm{r}^2-\omega_0^2)^2+4\alpha^2\omega_\mathrm{r}^2]^{1/2}}V_\mathrm{i}, \qquad \theta_\mathrm{o} = \theta_\mathrm{i} - \tan^{-1}[2\alpha\omega_0/(\omega_\mathrm{r}^2-\omega_0^2)]. \tag{51}$$

A point to note is that when $\alpha \ll \omega_r$, as is the case in a good ocillator, and the driving frequency ω_0 is close to the driving frequency, then V_out will be much larger in magnituce than V_in and almost exactly $\pi/2$ out of phase. Most of the energy at steady state is sloshing back and forth between the capacitor and the inductor, with a small amount added in on each cycle to compensate for the loss in the resistor.

An example of a transient solution can be found in the next section on oscillators.

Exercises

These exercises combine investigative exercises that require you to do some on-line research (I), mathematical exercises to fill in missing steps or provide additional examples (M), and computational exercises to illustrate functions in the complex domain (C).

1. (I) We made use of Taylor expansions in our derivations. How do you prove that and what the conditions for convergence of a Taylor expansion?

2. (I) We made use of the fundamental theorem of calculus to find the integrals of the cosine and sine functions. What is this theorem and how do you prove it?

3. (M) Show that the Taylor expansion for $\exp(z)$ converges for any complex z. Infer that the same is true for $\cos(z)$ and $\sin(z)$.

4. (M) We defined the exponential function using the definitions of the sine and cosine functions. A more usual way to define the exponential function is as

$$exp(x) = \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n.$$

   (a) Using this definition, derive the expressions for the Taylor expansion of $\exp(x)$ and $d\exp(x)/dx$. Show that $\exp(x+y) = \exp(x)\exp(y)$.

   (b) Use the complex exponential to define the cosine and sine functions, i.e., $\exp(i\theta) = \cos\theta + i\sin\theta$. Derive the usual expressions for the Taylor expansions and derivatives. Derive the angle additional formulae. Show that they are periodic with a period equal to 2π.

5. (M) Another way to define the exponential is as a function that satisfies the relation $f(x+y) = f(x)f(y)$, along with the condition that $\lim_{\Delta x \to 0} f(\Delta x) = 1 + \Delta x$. Show that you can derive the usual properties of the exponential function from this definition.

6. (M,C) The complex number z has n roots, $z^{1/n}$. Where are those roots located on the complex plane. If we $z = 3+3i$, plot the roots for $n = 9$.

7. (C) The close connection between the exponential function $\exp x$ and the sinusoidal functions $\cos x$ and $\sin x$ may appear a bit surprising at first since they look very different on the real axis. To gain some insight into the connection, we may plot their contours in the complex plane.

   (a) Use Matlab to plot contours of the real and imaginary parts of $\cos z$ and $\sin z$ where $x = \Re(z)$ varies between -2π and 2π, while $y = \Im(z)$ varies between -3 and 3.

   (b) Use Matlab to plot contours of the real and imaginary parts of $\exp(z)$, where $x = \Re(z)$ varies between -3 and 3, while $\Im(z)$ varies between -2π and 2π.

8. (C) Same as exercise 6 except for plotting the complex amplitude and phase of each function. [Hint: Watch out for the 2π jumps!]

9. (I) Section D reviews the portion of the theory of ordinary differential equations that is needed to understand oscillators (both mechanical and electrical). The textbook for MATH225 gives a far more detailed description of the applicable theory of ordinary equations. For example, in spring 2020, the textbook was *Introduction to Differential Equations* by S. Farlow; see chapter 3.

   (a) Discuss how our examples fall within the general framework of the theory of second-order homogeneous and non-homogeneous ordinary differential equations that is discussed in a standard undergraduate textbook on the theory of ordinary differential equations (your choice).

   (b) Discuss how the ideas that apply to second-order systems are extended to higher-order systems.

10. (M,I) We formulated the theory of linear ordinary differential equations using a single higher-order equation. It is usually more useful to formulate the theory as a coupled set of first-order equations, so that

$$\begin{aligned}
\frac{du_1}{dt} &= a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n, \\
\frac{du_2}{dt} &= a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n, \\
&\vdots \\
\frac{du_n}{dt} &= a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n,
\end{aligned} \tag{E9.1}$$

for a homogeneous (non-driven) system. This equation can be written in matrix form as

$$\frac{d\mathbf{u}}{dt} + \mathsf{A}\mathbf{u} = 0, \tag{E9.2}$$

where $\mathbf{u}$ and A, where $\mathbf{u}$ is an n-dimensional column vector and A an $n \times n$-dimensional matrix.

(a) (M) Show that Eq. (23) can be written in this form. [Hint: the derivatives can be used to define the u_n.]

(b) (I) When searching for the natural frequencies, we set $\mathbf{u} = \bar{\mathbf{u}} \exp(\gamma_j t)$, where $\mathbf{u}$ is a constant column vector. Equation E9.2 now becomes

$$(\mathsf{A} - \gamma \mathsf{I})\bar{\mathbf{u}} = 0, \tag{E9.2}$$

where I is the $n \times n$ identity matrix, which is the matrix that has 1 in all the diagonal elements and zero everywhere else. The condition for this equation to have solutions is for the determinant to equal zero. Give the definition of the determinant and show it corresponds to a polynomial of order n with real coefficients. The γ_j are referred to as *eigenvalues* or *characteristic values* of the system.

11. (M) Fill in the algebraic details between Eq. (29) and Eq. (38).

12. (M) Fill in the algebraic details between Eq. (45) and Eq. (50).

13. (M) The circuit that we considered in Fig. 6 and is expressed in Eq. (29) can be studied most efficiently by expressing it as two coupled first-order equations. Introducing the phasor domain representations for both $V(t)$ and $I(t)$, $V(t) = \tilde{V} \exp(\gamma t)$ and $I(t) = \tilde{I} \exp(\gamma t)$, find the two coupled algebraic equations that govern $\tilde{V}$ and $\tilde{I}$. Show that the values for γ are the same as we found before.

14. (M) The driven system of equations that we considered in Fig. 7 and is expressed in Eq. (45) can also be studied most efficiently by expressing it as two first-order equations and introducing the phasor domain representations for these equations. Use this representation to find the stead-state solutions.

15. (C) Typical values for R, L, and C in an oscillator circuit would be $R = 1\ \Omega$, $L = 100$ nH, and $C = 10$ pF. Assuming that the initial value of the voltage in the circuit is 1 V, plot $I(t)$ and $V(t)$.

16. (M) If the circuit that is represented in Fig. 7 and in Eq. (45) is turned on at $t = 0$, there will be a transient solution before the circuit reaches its steady state.

(a) Show that the initial condition is

$$V_{\text{out}}(t = 0) = 0, \qquad \left. \frac{dV_{\text{out}}}{dt} \right|_{t=0} = 0.$$

(b) Find the transient solution and the total solution.